Central Coverage Bayes Prediction Intervals for the Generalized Pareto Distribution

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Abstract

The Generalized Pareto model is considered underlying model from which observables are to be predicted by Bayesian approach. The Type–II censored data from the model is considered for the Central Coverage Bayes prediction technique. Both the known and unknown cases of the parameters have been considered here. A simulation study also has been carried out for illustrating the performances of the procedures.

Keywords

Generalized Pareto Model; Central Coverage Bayes Prediction; Type–II Censoring

Introduction

The Pareto model plays an important role in socio–economic studies. It is often used as a model for analysing areas including city population distribution, stock price fluctuation, oil field locations and military areas. It has been found to be suitable for approximating the right tails of distribution with positive skewness. The Pareto distribution and their close relatives provide a very flexible family of fat–tailed distributions, which may be used as a model for the income distribution of higher income group. The Pareto distribution has a decreasing failure rate, so it has often been used for model survival after some medical procedures (the ability to survive for a longer time appears to increase, the longer one survives after certain medical procedures).

An important objective of a life-testing experiment is to predict the nature of future sample based on a current sample. Prediction of mean, the smallest or the largest observation in a future sample, has been a topic of interest and importance in the context of quality and reliability analysis.

The objective of the present paper is to predict the nature of the future behaviour of the observation when sufficient information of the past and the present behaviour of an event or an observation is known or given for the generalized Pareto model. Under Type–II censoring scheme we obtain the Central Coverage Bayes prediction length of intervals. Both known and unknown case for the parameter has been considered.

A good deal of literature is available on predictive inference for future failure distribution under different criterion. Few of those who have been extensively studied predictive inference for the future observations for the Pareto model are Arnold & Press (1989), Ouyang & Wu (1994), Mousa (2001), Soliman (2001), Nigm et al. (2003), Wu et al. (2004), Raqab et al. (2007).

Model and Prior Distributions

The probability density function and cumulative density function of the considered generalized Pareto model are given as

$$f(x;\sigma,\theta)=\frac{1}{\sigma\theta}\left(1-\frac{x}{\sigma}\right)^{\frac{1}{\theta}-1};0<x<\sigma,\theta>0, \quad (2.1)$$

$$F(x;\sigma,\theta)=1-\left(1-\frac{x}{\sigma}\right)^{\frac{1}{\theta}}. \quad (2.2)$$

Here, θ is known as the shape parameter and σ as the scale parameter.

Suppose n items are put to test under model (2.1) without replacement and test terminates as soon as the r^{th} item fails $(r\le n)$. If $x_1,x_2,...,x_r$ be the observed failure items for the first r components, then the likelihood function for first r failure items $\underline{x}(=x_1,x_2,...,x_r)$ is

$$L(\underline{x}\,|\,\theta)\propto\theta^{-r}\sigma^{-r}\exp\left(-\frac{T_r}{\theta}-T_0\right), \quad (2.3)$$

where

$$T_r = -\left\{\sum_{i=1}^{r} \log\left(1-\frac{x_{(i)}}{\sigma}\right) + (n-r)\log\left(1-\frac{x_{(r)}}{\sigma}\right)\right\}$$

and $T_0 = \log\sum_{i=1}^{r}\left(1-\frac{x_{(i)}}{\sigma}\right)$.

When the shape parameter σ is known, the scale parameter θ has a conjugate prior density and in present case it is taken as inverted Gamma distribution with the probability density function

$$g(\theta) = \frac{\beta^{\alpha} e^{-\beta/\theta}}{\theta^{\alpha+1}\Gamma(\alpha)} \; ; \; \theta > 0, \alpha > 0, \beta > 0. \qquad (2.4)$$

We believe, as stated in Arnold and Press (1983) that from a Bayesian view point, there is clearly no way in which one can say that one prior is better than other. It is more frequently the case that, we select to restrict attention to a given flexible family of priors, and we choose one from that family, which seems to match best with our personal beliefs. Also the main reason for its general acceptability is the mathematical tractability resulting from the fact that the inverted gamma distribution is the conjugate prior for the scale parameter θ . Now, the corresponding posterior density for the parameter θ is obtained as

$$\pi(\theta) = \frac{(T_r+\beta)^{r+\alpha}}{\Gamma(r+\alpha)} \frac{\exp(-(T_r+\beta)/\theta)}{\theta^{r+\alpha+1}}. \qquad (2.5)$$

In most practical applications with proper informative information on the Pareto scale parameter, the prior variance is usually finite. Now we provide the prior information in the case when the shape parameter is unknown, which is more common in practice. It is known that in this case the Pareto distribution does not have a continuous conjugate joint prior distribution, although there exist a continuous-discrete joint prior distribution. The continuous component of this distribution is related to the scale parameter, and the discrete one is related to the shape parameter.

In present case when the parameter σ is unknown, the joint prior density for both the parameters θ and σ is taken as

$$g_1(\theta,\sigma) = \left(\frac{\beta^{\alpha} e^{-\beta/\theta}}{\theta^{\alpha+1}\Gamma(\alpha)}\right)\left(\frac{1}{\varphi}\right);$$

$$\theta > 0, 0 \le \sigma \le \varphi, (\alpha, \beta, \varphi) > 0. \qquad (2.6)$$

The joint posterior density for the considered parameters θ and σ is obtained as

$$\pi_1(\theta,\sigma) = \frac{\theta^{-(r+\alpha+1)} e^{-(T_r+\beta)/\theta} e^{-T_0}}{\overline{\sigma}\,\Gamma(r+\alpha)\,\sigma^{r}} \; ;$$

$$\overline{\sigma} = \int_{\sigma=0}^{\varphi} e^{-T_0} \frac{\sigma^{-r}}{(T_r+\beta)^{r+\alpha}} d\sigma. \qquad (2.7)$$

Central Coverage Bayes Prediction Interval (Parameter σ is Known)

Let $x_1, x_2, ..., x_r$ be the first r observed failure items from a sample of size n under the Type–II censoring scheme from the model (2.1). If $Y = (y_1, y_2, ..., y_m)$ be the second independent random sample of future observations from the same model. Then the Bayes predicative density of the future observation Y is denoted by $h(y\,|\,\underline{x})$ and obtained by simplifying

$$h(y\,|\,\underline{x}) = \int_{\theta} f(y;\sigma,\theta)\,\pi(\theta)\,d\theta$$

$$\Rightarrow h(y\,|\,\underline{x}) = \left(\frac{r+\alpha}{\sigma-y}\right) \frac{(T_r+\beta)^{r+\alpha}}{(T_r+\beta-\log(\sigma-y)+\log\sigma)^{r+\alpha+1}}. \qquad (3.1)$$

In the context of Bayes prediction, we say that (l_1, l_2) is a $100(1-\varepsilon)\%$ prediction limits for the future random variable Y, if

$$\Pr(l_1 < Y < l_2) = 1-\varepsilon. \qquad (3.2)$$

Here l_1 and l_2 are said to be lower and upper Bayes prediction limits for the random variable Y and $1-\varepsilon$ is called the confidence prediction coefficient. The Central coverage Bayes prediction lower and upper limits are obtain by solving following equality

$$\Pr(Y \le l_{C1}) = \frac{1-\varepsilon}{2} = \Pr(Y \ge l_{C2}). \qquad (3.3)$$

Here, l_{C1} and l_{C2} are said to be lower and upper Central coverage Bayes prediction limits for the random variable Y. Using (3.1) and (3.3), the lower and upper Bayes prediction limits of Y are obtained as

$$l_{C1} = \sigma \left\{ 1 - e^{(T_r+\beta)(1-\varepsilon_1)} \right\}; \ \varepsilon_1 = \left(\frac{1+\varepsilon}{2} \right)^{-1/(r+\alpha)} \quad (3.4)$$

and

$$l_{C2} = \sigma \left\{ 1 - e^{(T_r+\beta)(1-\varepsilon_2)} \right\}; \ \varepsilon_2 = \left(\frac{1-\varepsilon}{2} \right)^{-1/(r+\alpha)} \quad .(3.5)$$

Therefore, the Central coverage Bayes prediction length of interval for the given future observation Y is

$$I_C = l_{C2} - l_{C1}. \quad (3.6)$$

Numerical Analysis

To assess and study the properties of the Bayes prediction length of interval under the Central coverage the random samples are generated as follows:

1. For the purposefully given values of the prior parameters α and β, generate θ by using the prior density $g(\theta)$. The considered values of prior parameters are $(\alpha, \beta) =$ $(02, 0.50)$, $(04, 0.25)$ and $(08, 0.125)$.

   Here smaller values were consider for the prior parameters to reflect little prior information whereas larger reflect the higher prior information. Decisively, we kept the prior means the same as the original means, although we have seen (not reported here) that the overall behaviour remains the same.

2. Using θ obtained in step (1), and the considered values of $\sigma = 1.50, 1.00, 0.50$ generates the 10,000 random samples of size $n(=10)$ from the considered model (2.1).

3. The selected set of censored sample size $r = 04, 06,$ $08, 10$ with the level of significance $\varepsilon = 99\%,$ $95\%, 90\%$; the Central coverage Bayes prediction length of intervals has been obtained and presented in Table 01.

4. We observe from the table that the length of the interval tend to be closer as r increases or σ decrease when other parametric values are fixed. Similar trend has also been seen when combination of the prior parameter increases. It is also noted that when confidence level decreases the length of intervals also decreases.

Remark:

In the case when the censored sample size $r(=10)$, the censoring criterion is reduces to the complete sample size criterion and hence the result are valid for complete sample case.

Central Coverage Bayes Prediction Interval (Parameter σ is Unknown)

When both parameters are considered to be unknown, the Bayes predicative density for the future observation Y is denoted by $h_1(y|\underline{x})$ and obtained by simplifying

$$h_1(y|\underline{x}) = \int_\sigma \int_\theta f(y; \sigma, \theta)\, \pi_1(\theta, \sigma)\, d\theta\, d\sigma$$

$$\Rightarrow h_1(y|\underline{x}) = \frac{r+\alpha}{\overline{\sigma}}$$

$$\int_\sigma \frac{\left(T_r + \beta - \log(\sigma - y) + \log\sigma \right)^{-(r+\alpha+1)}}{\sigma^r e^{T_0} (\sigma - y)} d\sigma. \quad (5.1)$$

Using (5.1) and (3.3), the lower and upper Bayes prediction limits for Y are does obtained by solving following equations

$$\left(\frac{1-\varepsilon}{2} \right) \overline{\sigma} = \int_\sigma \frac{e^{-T_0}}{\sigma^r (T_r+\beta)^{r+\alpha}} \left\{ 1 - \left(1 - (T_r+\beta)^{-1} \log\left(1 - \frac{l_{C1}}{\sigma} \right) \right)^{-r-\alpha} \right\} d\sigma$$

and

$$\left(\frac{1+\varepsilon}{2} \right) \overline{\sigma} = \int_\sigma \frac{e^{-T_0}}{\sigma^r (T_r+\beta)^{r+\alpha}} \left\{ 1 - \left(1 - (T_r+\beta)^{-1} \log\left(1 - \frac{l_{C2}}{\sigma} \right) \right)^{-r-\alpha} \right\} d\sigma.$$

For the future observation Y, the Central coverage Bayes prediction length of the interval is

$$I_C = l_{C2} - l_{C1}. \quad (5.2)$$

Numerical Analysis

When both parameters considered as the random variable, a simulation study also has been carried out to study the properties of the Bayes prediction interval as follows:

1. For the similar set of the considered values of the prior parameters α and β, as taken in Section 4, the values of θ has been generated.
2. For the given values of prior parameter $\varphi\,(=0.50, 1.00, 5.00)$, generate a random value of σ from the uniform $U\left(0, 1/\varphi\right)$.
3. Using the above generated values of θ and σ obtained in steps (1) & (2) we generates the 10,000 random samples of size $n(=10)$ from the considered model (2.1).
4. For the similar set of selected values for r and ε (Section 4), the Bayes prediction lengths of intervals have been obtain and presented them in the Table 02.
5. It has been observed from the Table 02 that, all the properties of the Bayes prediction length of intervals are similar to the case when one parameter is known.

Conclusions

The Generalized Pareto model is considered here from the observables is to be predicted by Bayesian approach. The Type–II censored data is considered for the Central Coverage Bayes prediction technique under both known and unknown cases of the parameters. For illustrating the performances of the procedures a simulation study has been carried out under the purposefully choosing the values of the hyper parameters.

The conjugate prior for the scale parameter is considered here on the basis of the general acceptability. In case when the shape parameter is unknown (a more common in practice), a continuous-discrete joint prior distribution is considered.

For all the considered values it is observed that the central coverage Bayes prediction lengths of intervals are shorter. Also, the length tends to be shortest with decreasing in the sample size or the confidence level.

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TABLE 1 BAYES PREDICTION LENGTH OF THE INTERVAL WHEN σ IS KNOWN

		$\sigma = 1.50$			$\sigma = 0.50$			$\sigma = 0.25$		
r	$\alpha, \beta \downarrow$	99%	95%	90%	99%	95%	90%	99%	95%	90%
04	2, 0.50	5.2253	3.6653	1.8888	4.5719	1.0812	0.5431	0.8952	0.5427	0.2741
	4, 0.25	4.2984	1.3686	1.2375	1.0558	0.4996	0.4139	0.3688	0.2891	0.2580
	8, 0.125	1.6773	1.2530	0.5750	0.5042	0.2613	0.2314	0.2639	0.2465	0.2140
06	2, 0.50	4.4177	3.1275	1.5639	1.5260	0.6585	0.3658	0.5901	0.4772	0.2438
	4, 0.25	2.1640	1.3012	0.6280	0.5372	0.4970	0.3115	0.2559	0.1943	0.1695
	8, 0.125	0.9968	0.7898	0.4677	0.4338	0.1640	0.1387	0.1589	0.1282	0.1250
08	2, 0.50	3.2107	1.2136	1.0953	0.8955	0.4105	0.3192	0.3068	0.2140	0.1649
	4, 0.25	2.1192	0.9216	0.5288	0.4574	0.4039	0.2396	0.2253	0.1657	0.1574
	8, 0.125	0.7699	0.5345	0.3577	0.3570	0.1570	0.1334	0.1563	0.1260	0.1114
10	2, 0.50	2.4252	0.9805	0.7441	0.6816	0.3855	0.1954	0.2397	0.1887	0.1618
	4, 0.25	1.9542	0.8173	0.5233	0.3541	0.2987	0.1707	0.1876	0.1646	0.1429
	8, 0.125	0.7754	0.3747	0.3278	0.2827	0.1296	0.1196	0.1537	0.1259	0.1123

TABLE 2 BAYES PREDICTION LENGTH OF THE INTERVAL WHEN σ IS UNKNOWN

		$\varphi = 0.50$			$\varphi = 1.00$			$\varphi = 5.00$		
r	$\alpha, \beta \downarrow$	99%	95%	90%	99%	95%	90%	99%	95%	90%
04	2, 0.50	4.2597	3.7399	1.9477	3.6475	1.1146	0.5658	0.8853	0.5424	0.2768
	4, 0.25	3.3237	1.3965	1.2761	1.0771	0.5151	0.4312	0.3647	0.2889	0.2606
	8, 0.125	1.6937	1.2785	0.5929	0.5143	0.2694	0.2411	0.2609	0.2463	0.2162
06	2, 0.50	3.3908	3.1517	1.5925	1.5378	0.6704	0.3763	0.5764	0.4710	0.2431
	4, 0.25	2.1584	1.3112	0.6395	0.5413	0.5060	0.3205	0.2499	0.1918	0.1691
	8, 0.125	0.9942	0.7959	0.4763	0.4371	0.1669	0.1427	0.1553	0.1265	0.1247
08	2, 0.50	3.2356	1.2359	1.1272	0.9117	0.4223	0.3319	0.3028	0.2134	0.1663
	4, 0.25	2.1357	0.9385	0.5442	0.4657	0.4155	0.2492	0.2224	0.1652	0.1587
	8, 0.125	0.7758	0.5444	0.3681	0.3635	0.1615	0.1387	0.1543	0.1257	0.1123
10	2, 0.50	2.4142	0.9862	0.7563	0.6855	0.3917	0.2007	0.2337	0.1858	0.1610
	4, 0.25	1.9452	0.8220	0.5318	0.3560	0.3035	0.1753	0.1829	0.1622	0.1422
	8, 0.125	0.7718	0.3769	0.3331	0.2844	0.1316	0.1228	0.1498	0.1240	0.1118